Numerical Explorations for Fast Spectrum of Fractional Gaussian Noise

Martin Mächler
ETH Zurich
June 2011 (LaTeX'ed October 9, 2024)

Abstract

The package **longmemo** ....
Paxson (1997) ... ...

Keywords: Euler-Maclaurin Formula, Fractional Gaussian Noise, Spectrum.

1. .. intro ..

The spectral density of fractional Gaussian noise ("fGn") with Hurst parameter $H \in (0,1)$ is (Beran (1986, 1994))

$$f_H(\lambda) = \mathcal{A}(\lambda, H)\left(|\lambda|^{-2H-1} + \mathcal{B}(\lambda, H)\right), \tag{1}$$

for $\lambda \in [-\pi, \pi]$, where $\mathcal{A}(\lambda, H) == 2\sin(\pi H)\,\Gamma(2H+1)\,(1-\cos\lambda)$, and

$$\mathcal{B}(\lambda, H) = \sum_{j=1}^{\infty}\left((2\pi j + \lambda)^{-(2H+1)} + (2\pi j - \lambda)^{-(2H+1)}+\right). \tag{2}$$

For the Whittle estimator of H and also other purposes, its advantageous to be able to evaluate $f_H(\lambda_i)$ efficiently for a whole vector of λ_i, typically Fourier frequencies $\lambda_i = 2\pi i/n$, for $i = 1, 2, \dots, \lfloor (n-1)/2 \rfloor$. Such evaluation is problematic because of the infinite sum for $\mathcal{B}(\lambda, H)$ in (2).

Traditionally, e.g., already in Appendix.... of Beran (1994), the infinite sum $\sum_{j=1}^{\infty}$ had been replaced by $\sum_{j}^{200}$ — which was still not very efficient and not extremely accurate. In our R package **longmemo**, we now provide the function `B.specFGN`(λ, H) to compute $\mathcal{B}(\lambda, H)$, using several ways to compute the infinite sum approximately, e.g., for $H = 0.75$ and $n = 500$, i.e., at 250 Fourier frequencies,

```
> require("longmemo")
> fr <- .ffreq(500)
> B.1   <- B.specFGN(fr, H = 0.75, nsum =   200, k.approx=NA)
> B.xct <- B.specFGN(fr, H = 0.75, nsum = 10000, k.approx=NA)
> all.equal(B.xct, B.1)
```

```
[1] "Mean relative difference: 0.0001243095"
```

which means that the 200 term approximation is accurate to 4 decimal digits for $H = .75$ but the accuracy is smaller for smaller H.

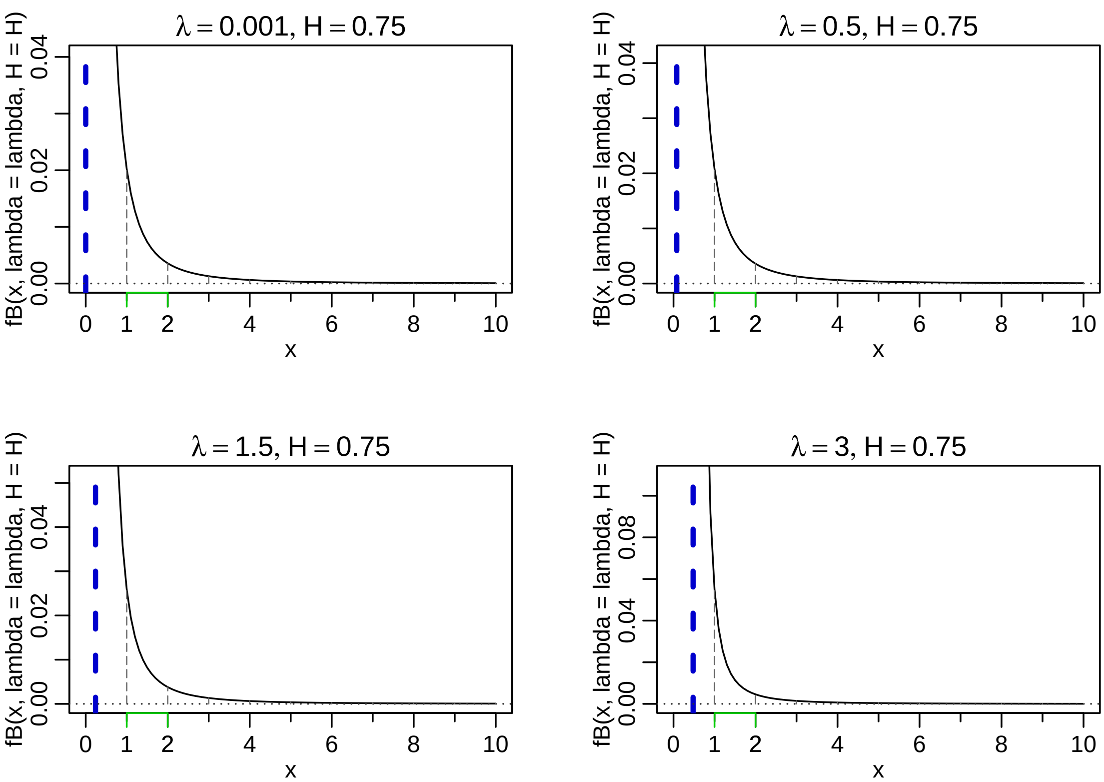
For this reason, Paxson (1997) derived formulas for fast and stilly quite accurate approximations of $\mathcal{B}(\lambda, H)$, noting that $\mathcal{B}(\lambda, H) = \sum_{j=1}^{\infty} f(j; \lambda, H)$ for

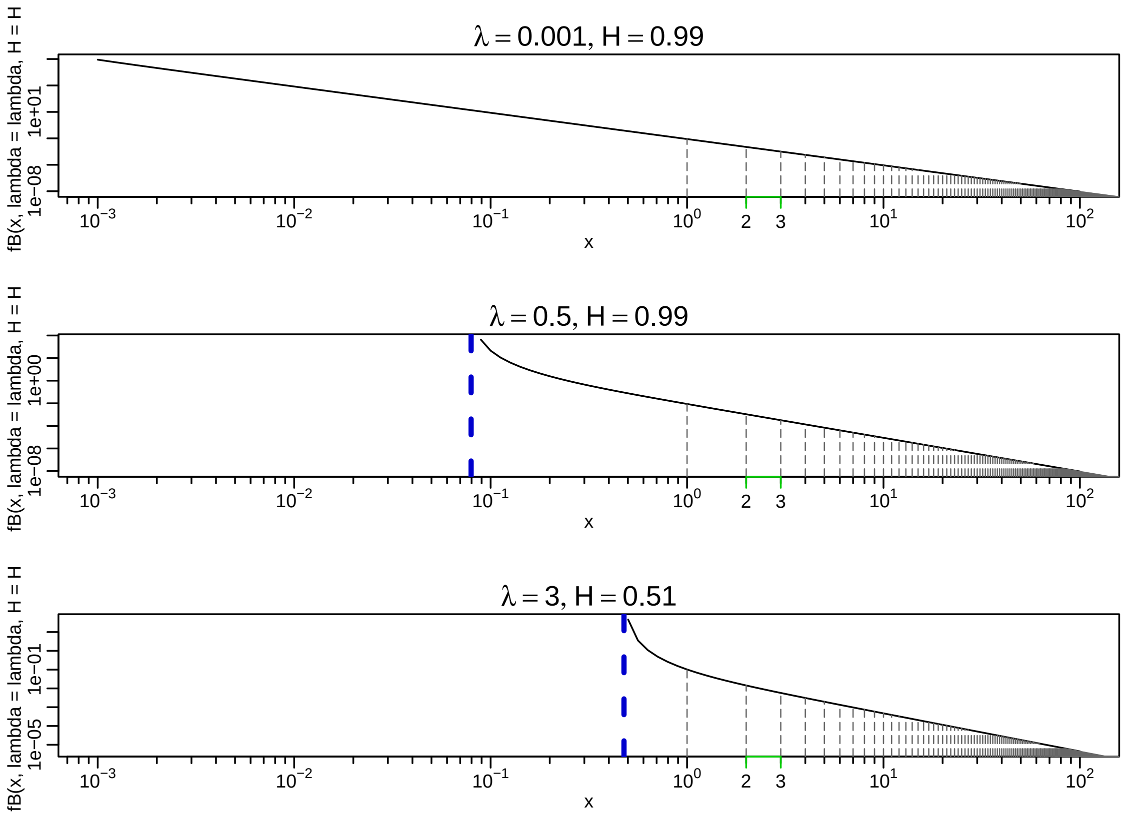
$$f(x; \lambda, H) = (2\pi x + \lambda)^{-(2H+1)} + (2\pi x - \lambda)^{-(2H+1)}, \tag{3}$$

and the fact that $\sum_{j=1}^{\infty} f(j)$ is a Riemann sum approximation of $\int_0^{\infty} f(x)\, \mathrm{d}x$ or $\int_1^{\infty} f(x)\, \mathrm{d}x$.

```
> fB <- function(x, lambda, H) {
      u <- 2 * pi * x
      h <- -(2 * H + 1)
      (u + lambda)^h + (u - lambda)^h
  }
```

Now its clear that $f(x)$ cannot be computed (or "is infinite") at $x = 0$, and more specifically, $f(x)$ tends to ∞ when $x \to \frac{\lambda}{2\pi}$, as in the second term of f, $2\pi x - \lambda$ only remains positive when $2\pi x > \lambda$. This is always fulfilled for $xin\{1, 2, \dots\}$, as $\lambda < \pi$, but is problematic when considering $\int_0^b f(x)\, \mathrm{d}x$ as above. Some illustrations of the function $f(x; \lambda, H)$ and its "pole" at $\frac{\lambda}{2\pi}$:





So, very clearly, Paxson's first formula, using $\int_0^1 f(x)\,\mathrm{d}x$ is not feasible, as $f(x)$ is *not* defined (or defined as ∞) for $x \le \lambda/(2\pi)$.

However, his generalized formula, "(7), p. 15",

$$\sum_{i=1}^{\infty} f_i \approx \sum_{j=1}^{k} f_j + \frac{1}{2}\int_k^{k+1} f(x)\,\mathrm{d}x + \int_{k+1}^{\infty} f(x)\,\mathrm{d}x, \tag{4}$$

clearly *is* usable for $k \ge 1$ (but not for $k = 0$, contrary to what he suggests). Indeed, with `B.specFGN(`λ, H, `k.approx)`, we now provide the result of applying approximation (4) to the infinite sum for $\mathcal{B}(\lambda, H)$ in (2).

Paxson ended the $k = 3$ approximation which he further improved considerably, empirically, by numerical comparison (and least squares fitting) with the "accurate" formula using `nsum =` $10'000$ terms. In the following section, we propose another improvement over Paxson's original idea:

2. Better approximations using the Euler–Maclaurin formula

Copied straight from `http://en.wikipedia.org/wiki/Euler-Maclaurin_formula` :

If n is a natural number and $f(x)$ is a smooth, i.e., sufficiently often differentiable function defined for all real numbers x between 0 and n, then the integral

$$I = \int_0^n f(x)\,dx \tag{5}$$

can be approximated by the sum (or vice versa)

$$S = \frac{1}{2}f(0) + f(1) + \cdots + f(n-1) + \frac{1}{2}f(n)$$

(see trapezoidal rule). The Euler–Maclaurin formula provides expressions for the difference between the sum and the integral in terms of the higher derivatives $f^{(k)}$ at the end points of the interval 0 and n. Explicitly, for any natural number p, we have

$$S - I = \sum_{k=2}^{p} \frac{B_k}{k!} \left(f^{(k-1)}(n) - f^{(k-1)}(0) \right) + R$$

where $B_1 = -1/2$, $B_2 = 1/6$, $B_3 = 0$, $B_4 = -1/30$, $B_6 = 1/42$, $B_8 = -1/30$, $\ldots$ are the Bernoulli numbers, and R is an error term which is normally small for suitable values of p. (The formula is often written with the subscript taking only even values, since the odd Bernoulli numbers are zero except for B_1.)

Note that

$$-B_1(f(n) + f(0)) = \frac{1}{2}(f(n) + f(0)).$$

Hence, we may also write the formula as follows:

$$\sum_{i=0}^{n} f(i) = \int_0^n f(x)\, dx - B_1(f(n) + f(0)) + \sum_{k=1}^{p} \frac{B_{2k}}{(2k)!} \left(f^{(2k-1)}(n) - f^{(2k-1)}(0) \right) + R. \quad (6)$$

....

In the context of computing asymptotic expansions of sums and series, usually the most useful form of the Euler–Maclaurin formula is

$$\sum_{n=a}^{b} f(n) \sim \int_a^b f(x)\, dx + \frac{f(a) + f(b)}{2} + \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} \left(f^{(2k-1)}(b) - f^{(2k-1)}(a) \right),$$

where a and b are integers. Often the expansion remains valid even after taking the limits $a \to -\infty$ or $b \to +\infty$, or both. In many cases the integral on the right-hand side can be evaluated in closed form in terms of elementary functions even though the sum on the left-hand side cannot.

(end of citation from Wikipedia)

— —

3. Session Information

```
> toLatex(sessionInfo())
```

- R version 4.4.1 (2024-06-14), `x86_64-pc-linux-gnu`
- Locale: `LC_CTYPE=en_US.UTF-8`, `LC_NUMERIC=C`, `LC_TIME=en_US.UTF-8`, `LC_COLLATE=C`, `LC_MONETARY=en_US.UTF-8`, `LC_MESSAGES=C`, `LC_PAPER=en_US.UTF-8`, `LC_NAME=C`, `LC_ADDRESS=C`, `LC_TELEPHONE=C`, `LC_MEASUREMENT=en_US.UTF-8`, `LC_IDENTIFICATION=C`

- Time zone: `Etc/UTC`
- TZcode source: `system (glibc)`
- Running under: `Ubuntu 24.04.1 LTS`
- Matrix products: default
- BLAS: `/usr/lib/x86_64-linux-gnu/openblas-pthread/libblas.so.3`
- LAPACK: `/usr/lib/x86_64-linux-gnu/openblas-pthread/libopenblasp-r0.3.26.so` ; LAPACK version3.12.0
- Base packages: base, datasets, grDevices, graphics, methods, stats, utils
- Other packages: longmemo 1.1-3
- Loaded via a namespace (and not attached): buildtools 1.0.0, compiler 4.4.1, knitr 1.48, maketools 1.3.0, sfsmisc 1.1-19, sys 3.4.2, tools 4.4.1, xfun 0.47

4. Conclusion

References

Beran J (1986). *Estimation, testing and prediction for self-similar and related processes.* Ph. d. thesis, no 8074, Swiss Federal Institute of Technology (ETH).

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Affiliation:

Martin Mächler
Seminar für Statistik, HG G 16
ETH Zurich
8092 Zurich, Switzerland
E-mail: `maechler@stat.math.ethz.ch`
URL: `http://stat.ethz.ch/people/maechler`